Lecture 9

Waves in Gyrotropic Media, Polarization

We have studied TEM uniform plane wave in Lecture 7. When the **k** vector is pointing in the z direction for instance, the electric field is polarized in the xy plane. Assume that the electric field is polarized in the x axis, when such a wave propagates through a gyrotropic medium, it electric field rotates as it propagates as we shall see. It can be polarized in other directions after a propagation distance, such as the y direction. Therefore, gyrotropy is an important concept in electromagnetics. In general, when a wave propagates through a gyrotropic medium, the electric field rotates changing the polarization of the wave. Our ionosphere is such a medium, and it affects radio and microwave communications between the Earth and the satellite by affecting the polarization of the wave. We will study this important topic in this lecture, and the general polarization of waves.

9.1 Gyrotropic Media and Faraday Rotation

This section derives the effective permittivity tensor of a gyrotropic medium in the ionsphere. Our ionosphere is always biased by a static magnetic field due to the Earth's magnetic field [77]. But in this derivation, in order to capture the salient feature of the physics with a simple model, we assume that the ionosphere has a static magnetic field polarized in the z direction, namely that $\mathbf{B} = \hat{z}B_0$. Now, the equation of motion from the Lorentz force law for an electron with $q = -e$, (in accordance with Newton's second law that $F = ma$ or force equals mass times accelration) becomes

$$m_e \frac{d\mathbf{v}}{dt} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \qquad m_e \frac{d^2\mathbf{r}}{dt^2} = -e\left(\mathbf{E} + \frac{d\mathbf{r}}{dt} \times \mathbf{B}\right) \tag{9.1.1}$$

The first term of the force on the right-hand side is similar to Coulomb force, while the second term is usually termed the $\mathbf{v} \times \mathbf{B}$ force.[1]

[1]For a plane wave, it can be shown that the $\mathbf{v} \times \mathbf{B}$ force is of order v/c smaller than the Coulomb force, which is termed relativistically small.

Next, let us assume that the electric field is polarized in the xy plane. The derivative of $\mathbf{v}$ is the acceleration of the electron, and also, $\mathbf{v} = d\mathbf{r}/dt$ where $\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z$. Again, assuming linearity, we use frequency domain technique for the analysis. And in the frequency domain, the above equation in the cartesian coordinates becomes

$$m_e\omega^2 x = e(E_x + j\omega B_0 y) \tag{9.1.2}$$

$$m_e\omega^2 y = e(E_y - j\omega B_0 x) \tag{9.1.3}$$

The above constitutes two equations with two unknowns x and y. They cannot be solved easily for x and y in terms of the electric field because they correspond to a two-by-two matrix system with cross coupling between the unknowns x and y. But they can be simplified as follows: We can multiply (9.1.3) by $\pm j$ and add it to (9.1.2) to get two decoupled equations [78]:

$$m_e\omega^2(x + jy) = e[(E_x + jE_y) + \omega B_0(x + jy)] \tag{9.1.4}$$

$$m_e\omega^2(x - jy) = e[(E_x - jE_y) - \omega B_0(x - jy)] \tag{9.1.5}$$

In the above, if we take the new unknowns to be $x \pm jy$, the two equations are decoupled with respect to to these two unknowns. Defining new variables such that

$$s_\pm = x \pm jy \tag{9.1.6}$$

$$E_\pm = E_x \pm jE_y \tag{9.1.7}$$

then (9.1.4) and (9.1.5) become

$$m_e\omega^2 s_\pm = e(E_\pm \pm \omega B_0 s_\pm) \tag{9.1.8}$$

Thus, solving the above yields

$$s_\pm = \frac{e}{m_e\omega^2 \mp eB_0\omega} E_\pm = C_\pm E_\pm \tag{9.1.9}$$

where

$$C_\pm = \frac{e}{m_e\omega^2 \mp eB_0\omega} \tag{9.1.10}$$

(By this manipulation, the above equations (9.1.2) and (9.1.3) transform to new equations where there is no cross coupling between $s_\pm$ and $E_\pm$. The mathematical parlance for this is the diagonalization of a matrix equation [79]. Thus, the new equation can be solved easily.)

Next, one can define $P_x = -Nex$, $P_y = -Ney$, and that $P_\pm = P_x \pm jP_y = -Nes_\pm$. Then it can be shown that

$$P_\pm = \varepsilon_0 \chi_\pm E_\pm \tag{9.1.11}$$

The expression for $\chi_\pm$ can be derived, and they are given as

$$\chi_\pm = -\frac{NeC_\pm}{\varepsilon_0} = -\frac{Ne}{\varepsilon_0}\frac{e}{m_e\omega^2 \mp eB_o\omega} = -\frac{{\omega_p}^2}{\omega^2 \mp \Omega\omega} \tag{9.1.12}$$

where Ω and ω_p are the cyclotron frequency[2] and plasma frequency, respectively, viz.,

$$\Omega = \frac{eB_0}{m_e}, \quad {\omega_p}^2 = \frac{Ne^2}{m_e\varepsilon_0} \tag{9.1.13}$$

At the cyclotron frequency, $|\chi_\pm| \to \infty$. In other words, $P_\pm$ is finite even when $E_\pm = 0$, or a solution exists to the equation of motion (9.1.1) without a forcing term, which in this case is the electric field. Thus, at this frequency, the solution blows up if the forcing term, $E_\pm$ is not zero. This is like what happens to an LC tank circuit at resonance whose current or voltage tends to infinity when the forcing term, like the voltage or current is nonzero.

In order to derive the permittivity tensor in the cartesian coordinates, one needs to express the original variables $P_x,\ P_y,\ E_x,\ E_y$ in terms of $P_\pm$ and $E_\pm$. With the help of (9.1.11), we arrive at

$$\begin{aligned} P_x = \frac{P_+ + P_-}{2} = \frac{\varepsilon_0}{2}(\chi_+E_+ + \chi_-E_-) &= \frac{\varepsilon_0}{2}[\chi_+(E_x + jE_y) + \chi_-(E_x - jE_y)] \\ &= \frac{\varepsilon_0}{2}[(\chi_+ + \chi_-)E_x + j(\chi_+ - \chi_-)E_y] \end{aligned} \tag{9.1.14}$$

$$\begin{aligned} P_y = \frac{P_+ - P_-}{2j} = \frac{\varepsilon_0}{2j}(\chi_+E_+ - \chi_-E_-) &= \frac{\varepsilon_0}{2j}[\chi_+(E_x + jE_y) - \chi_-(E_x - jE_y)] \\ &= \frac{\varepsilon_0}{2j}[(\chi_+ - \chi_-)E_x + j(\chi_+ + \chi_-)E_y] \end{aligned} \tag{9.1.15}$$

The above relationship in cartesian coordinates can be expressed using a tensor where

$$\mathbf{P} = \varepsilon_0\overline{\boldsymbol{\chi}}\cdot\mathbf{E} \tag{9.1.16}$$

where $\mathbf{P} = [P_x, P_y]$, and $\mathbf{E} = [E_x, E_y]$. From (9.1.14) and (24.2.9) above, $\overline{\boldsymbol{\chi}}$ is of the form

$$\overline{\boldsymbol{\chi}} = \frac{1}{2}\begin{pmatrix} (\chi_+ + \chi_-) & j(\chi_+ - \chi_-) \\ -j(\chi_+ - \chi_-) & (\chi_+ + \chi_-) \end{pmatrix} = \begin{pmatrix} -\frac{{\omega_p}^2}{\omega^2-\Omega^2} & -j\frac{{\omega_p}^2\Omega}{\omega(\omega^2-\Omega^2)} \\ j\frac{{\omega_p}^2\Omega}{\omega(\omega^2-\Omega^2)} & -\frac{{\omega_p}^2}{\omega^2-\Omega^2} \end{pmatrix} \tag{9.1.17}$$

Notice that in the above, when the $\mathbf{B}$ field is turned off or $\Omega = 0$, then $\overline{\boldsymbol{\chi}}$ above is diagonalize, and it resembles an isotropic medium of a collisionless, cold plasma again.

Consequently, for the $\mathbf{B} \neq 0$ case, the above can be generalized to 3D to give

$$\overline{\boldsymbol{\chi}} = \begin{bmatrix} \chi_0 & j\chi_1 & 0 \\ -j\chi_1 & \chi_0 & 0 \\ 0 & 0 & \chi_p \end{bmatrix} \tag{9.1.18}$$

where $\chi_p = -\omega_p^2/\omega^2$. Notice that since we assume that $\mathbf{B} = \hat{z}B_0$, the z component of (9.1.1) is unaffected by the $\mathbf{v}\times\mathbf{B}$ force. Hence, the electron moving in the z is like that of a cold collisionless plasma.

Using the fact that $\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0(\bar{\mathbf{I}} + \overline{\boldsymbol{\chi}})\cdot\mathbf{E} = \overline{\boldsymbol{\varepsilon}}\cdot\mathbf{E}$, the above implies that

$$\overline{\boldsymbol{\varepsilon}} = \varepsilon_0\begin{bmatrix} 1+\chi_0 & j\chi_1 & 0 \\ -j\chi_1 & 1+\chi_0 & 0 \\ 0 & 0 & 1+\chi_p \end{bmatrix} \tag{9.1.19}$$

[2]This is also called the gyrofrequency.

Now, $\overline{\varepsilon}$ is that of an anisotropic medium, of which a gyrotropic medium belongs. Please notice that the above tensor is a hermitian tensor. We shall learn later that this is the hallmark of a lossless medium.

Another characteristic of a gyrotropic medium is that a linearly polarized wave will rotate when passing through it. This is the Faraday rotation effect [78], which we shall learn more later. This phenomenon poses a severe problem for Earth-to-satellite communication, using linearly polarized wave as it requires the alignment of the Earth-to-satellite antennas. This can be avoided using a rotatingly polarized wave, called a circularly polarized wave that we shall learn in the next section.

As we have learnt, the ionosphere affects out communication systems two ways: It acts as a mirror for low-frequency electromagnetic or radio waves (making the experiment of Marconi a rousing success). It also affects the polarization of the wave. But the ionosphere of the Earth and the density of electrons that are ionized is highly dependent on temperature, and the effect of the Sun. The fluctuation of particles in the ionosphere gives rise to scintillation effects due to electron motion and collision that affect radio wave communication systems [80].

9.2 Wave Polarization

Studying wave polarization is very important for communication purposes [33]. A wave whose electric field is pointing in the x direction while propagating in the z direction is called a linearly polarized (LP) wave . The same can be said of one with electric field polarized in the y direction. It turns out that a linearly polarized wave suffers from Faraday rotation when it propagates through the ionosphere. For instance, an x polarized wave can become a y polarized wave due to Faraday rotation. So its polarization becomes ambiguous as the wave propagates through the ionosphere: to overcome this, Earth to satellite communication is done with circularly polarized (CP) waves [81]. So even if the electric field vector is rotated by Faraday's rotation, it remains to be a CP wave. We will study these polarized waves next.

9.2.1 General Polarizations—Elliptical and Circular Polarizations

We can write a general uniform plane wave propagating in the z direction in the time domain for simplicity as

$$\mathbf{E} = \hat{x}E_x(z,t) + \hat{y}E_y(z,t) \tag{9.2.1}$$

Clearly, $\nabla \cdot \mathbf{E} = 0$, and $E_x(z,t)$ and $E_y(z,t)$, by the principle of linear superposition, are solutions to the one-dimensional wave equation. For a time harmonic field, the two components may not be in phase, and we have in general for time domain that

$$E_x(z,t) = E_1 \cos(\omega t - \beta z) \tag{9.2.2}$$

$$E_y(z,t) = E_2 \cos(\omega t - \beta z + \alpha) \tag{9.2.3}$$

where α denotes the phase difference between these two wave components. We shall study how the linear superposition of these two components behaves for different α's. First, we set $z = 0$ to observe this field. Then

$$\mathbf{E} = \hat{x}E_1 \cos(\omega t) + \hat{y}E_2 \cos(\omega t + \alpha) \tag{9.2.4}$$

For $\alpha = \frac{\pi}{2}$

$$E_x = E_1 \cos(\omega t),\ E_y = E_2 \cos(\omega t + \pi/2) \tag{9.2.5}$$

Next, we evaluate the above for different ωt's

$$\omega t = 0, \qquad E_x = E_1, \qquad E_y = 0 \tag{9.2.6}$$
$$\omega t = \pi/4, \qquad E_x = E_1/\sqrt{2}, \qquad E_y = -E_2/\sqrt{2} \tag{9.2.7}$$
$$\omega t = \pi/2, \qquad E_x = 0, \qquad E_y = -E_2 \tag{9.2.8}$$
$$\omega t = 3\pi/4, \qquad E_x = -E_1/\sqrt{2}, \qquad E_y = -E_2/\sqrt{2} \tag{9.2.9}$$
$$\omega t = \pi, \qquad E_x = -E_1, \qquad E_y = 0 \tag{9.2.10}$$

The tip of the vector field **E** traces out an ellipse as show in Figure 9.1. With the left-hand thumb pointing in the z direction, the direction of propagation, and the wave rotating in the direction of the fingers, such a wave is called left-hand elliptically polarized (LHEP) wave.

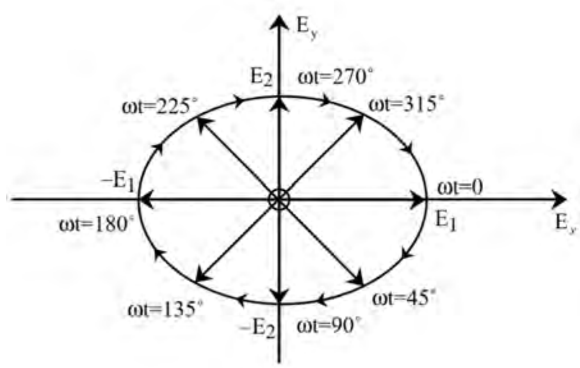

Figure 9.1: If one follows the tip of the electric field vector, it traces out an ellipse as a function of time t.

When $E_1 = E_2$, the ellipse becomes a circle, and we have a left-hand circularly polarized (LHCP) wave. When $\alpha = -\pi/2$, the wave rotates in the counter-clockwise direction, and the wave is either right-hand elliptically polarized (RHEP), or right-hand circularly polarized (RHCP) wave depending on the ratio of E_1/E_2. Figure 9.2 shows the different polarizations of the wave wave for different phase differences and amplitude ratio. Figure 9.3 shows a graphic picture of a CP wave propagating through space.

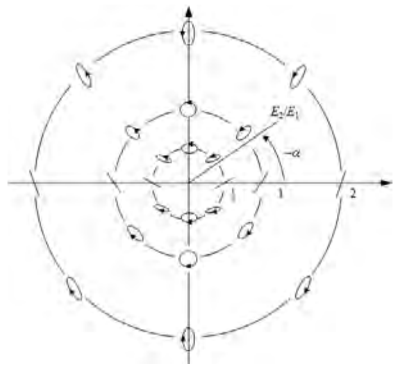

Figure 9.2: Due to different phase difference between the E_x and E_y components of the field, and their relative amplitudes E_2/E_1, different polarizations will ensure. The arrow indicates the direction of rotation of the field vector.

Figure 9.3: The rotation of the field vector of a right-hand circular polarization wave as it propagates in the right direction [82] (courtesy of Wikipedia).

9.2.2 Arbitrary Polarization Case and Axial Ratio[3]

As seen before, the tip of the field vector traces out an ellipse in space as it propagates. The axial ratio (AR) is the ratio of the major axis to the minor axis of this ellipse. It is an important figure of merit for designing CP (circularly polarized) antennas (antennas that will radiate circularly polarized waves). The closer is this ratio to 1, the better is the antenna design. We will discuss the general polarization and the axial ratio of a wave.

For the general case for arbitrary α, we let

$$E_x = E_1 \cos \omega t, \qquad E_y = E_2 \cos(\omega t + \alpha) = E_2(\cos \omega t \cos \alpha - \sin \omega t \sin \alpha) \tag{9.2.11}$$

[3]This section is mathematically complicated. It can be skipped on first reading.

Then from the above, expressing E_y in terms of E_x, one gets

$$E_y = \frac{E_2}{E_1} E_x \cos\alpha - E_2 \left[1 - \left(\frac{E_x}{E_1}\right)^2\right]^{1/2} \sin\alpha \tag{9.2.12}$$

Rearranging and squaring, we get

$$a{E_x}^2 - bE_xE_y + c{E_y}^2 = 1 \tag{9.2.13}$$

where

$$a = \frac{1}{{E_1}^2 \sin^2\alpha}, \quad b = \frac{2\cos\alpha}{E_1E_2\sin^2\alpha}, \quad c = \frac{1}{{E_2}^2\sin^2\alpha} \tag{9.2.14}$$

After letting $E_x \to x$, and $E_y \to y$, equation (9.2.13) is of the form,

$$ax^2 - bxy + cy^2 = 1 \tag{9.2.15}$$

The equation of an ellipse in its self coordinates is

$$\left(\frac{x'}{A}\right)^2 + \left(\frac{y'}{B}\right)^2 = 1 \tag{9.2.16}$$

where A and B are axes of the ellipse as shown in Figure 9.4. We can transform the above back to the (x, y) coordinates to get (9.2.15). To this end, we let

$$x' = x\cos\theta - y\sin\theta \tag{9.2.17}$$

$$y' = x\sin\theta + y\cos\theta \tag{9.2.18}$$

to get

$$x^2\left(\frac{\cos^2\theta}{A^2} + \frac{\sin^2\theta}{B^2}\right) - xy\sin 2\theta\left(\frac{1}{A^2} - \frac{1}{B^2}\right) + y^2\left(\frac{\sin^2\theta}{A^2} + \frac{\cos^2\theta}{B^2}\right) = 1 \tag{9.2.19}$$

Comparing (9.2.13) and (9.2.19), one gets

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2\cos\alpha E_1E_2}{{E_2}^2 - {E_1}^2}\right) \tag{9.2.20}$$

$$\mathrm{AR} = \left(\frac{1+\Delta}{1-\Delta}\right)^{1/2} > 1 \tag{9.2.21}$$

where AR is the axial ratio and

$$\Delta = \left(1 - \frac{4{E_1}^2{E_2}^2\sin^2\alpha}{\left({E_1}^2 + {E_2}^2\right)^2}\right)^{1/2} \tag{9.2.22}$$

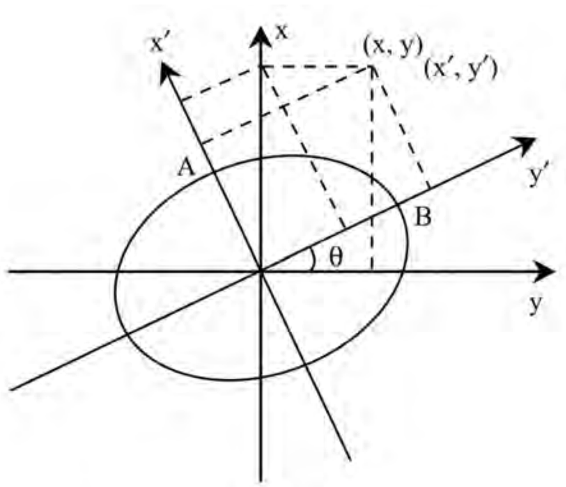

Figure 9.4: This figure shows the parameters used to derive the axial ratio (AR) of an elliptically polarized wave.

9.3 Polarization and Power Flow

For a linearly polarized wave in the time domain,

$$\mathbf{E} = \hat{x}E_0 \cos(\omega t - \beta z), \quad \mathbf{H} = \hat{y}\frac{E_0}{\eta} \cos(\omega t - \beta z) \tag{9.3.1}$$

Hence, the instantaneous power we have learnt previously in Section 5.3 becomes

$$\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t) = \hat{z}\frac{{E_0}^2}{\eta} \cos^2(\omega t - \beta z) \tag{9.3.2}$$

indicating that for a linearly polarized wave, the instantaneous power is function of both time and space. It travels as lumps of energy through space. In the above E_0 is the amplitude of the linearly polarized wave. Moreover, taking the time average of the above, we have

$$\langle \mathbf{S}(t) \rangle = \hat{z}\frac{{E_0}^2}{2\eta} \tag{9.3.3}$$

Next, we look at power flow for for elliptically and circularly polarized waves. It is to be noted that in the phasor world or frequency domain, (9.2.1) becomes

$$\mathbf{E}(z, \omega) = \hat{x}E_1 e^{-j\beta z} + \hat{y}E_2 e^{-j\beta z + j\alpha} \tag{9.3.4}$$

For LHEP wave,

$$\mathbf{E}(z, \omega) = e^{-j\beta z}(\hat{x}E_1 + j\hat{y}E_2) \tag{9.3.5}$$

whereas for LHCP wave,

$$\mathbf{E}(z,\omega) = e^{-j\beta z} E_1(\hat{x} + j\hat{y}) \tag{9.3.6}$$

For RHEP wave, the above becomes

$$\mathbf{E}(z,\omega) = e^{-j\beta z}(\hat{x}E_1 - j\hat{y}E_2) \tag{9.3.7}$$

whereas for RHCP wave, it is

$$\mathbf{E}(z,\omega) = e^{-j\beta z} E_1(\hat{x} - j\hat{y}) \tag{9.3.8}$$

Focussing on the circularly polarized wave,

$$\mathbf{E} = (\hat{x} \pm j\hat{y}) E_0 e^{-j\beta z} \tag{9.3.9}$$

Using that $\boldsymbol{\beta} = \hat{z}\beta$, and letting $\nabla \to -j\boldsymbol{\beta}$, Faraday's law becomes

$$\mathbf{H} = \frac{\boldsymbol{\beta} \times \mathbf{E}}{\omega\mu} \tag{9.3.10}$$

And then

$$\mathbf{H} = (\mp\hat{x} - j\hat{y}) j \frac{E_0}{\eta} e^{-j\beta z} \tag{9.3.11}$$

where $\eta = \sqrt{\mu/\varepsilon}$ is the intrinsic impedance of the medium. Therefore,

$$\mathbf{E}(t) = \hat{x}E_0\cos(\omega t - \beta z) \pm \hat{y}E_0\sin(\omega t - \beta z) \tag{9.3.12}$$

$$\mathbf{H}(t) = \mp\hat{x}\frac{E_0}{\eta}\sin(\omega t - \beta z) + \hat{y}\frac{E_0}{\eta}\cos(\omega t - \beta z) \tag{9.3.13}$$

Then the instantaneous power becomes

$$\mathbf{S}(t) = \mathbf{E}(t) \times \mathbf{H}(t) = \hat{z}\frac{{E_0}^2}{\eta}\cos^2(\omega t - \beta z) + \hat{z}\frac{{E_0}^2}{\eta}\sin^2(\omega t - \beta z) = \hat{z}\frac{{E_0}^2}{\eta} \tag{9.3.14}$$

In other words, a CP wave delivers constant instantaneous power independent of space and time, as opposed to a linearly polarized wave which delivers a non-constant instantaneous power as shown in (9.3.2). Moreover, taking the time average of the above, we have

$$\langle \mathbf{S}(t) \rangle = \hat{z}\frac{{E_0}^2}{\eta} \tag{9.3.15}$$

It is to be noted that the complex Poynting's vector for a lossless medium

$$\underset{\sim}{\mathbf{S}} = \mathbf{E} \times \mathbf{H}^* \tag{9.3.16}$$

is real and constant independent of space both for linearly, circularly, and elliptically polarized waves. That is if we were to go through the exercise to obtain $\underset{\approx}{\mathbf{S}}$ for the general case, we will let

$$\mathbf{E} = (\hat{x}E_1 \pm j\hat{y}E_2)e^{-j\beta z} \tag{9.3.17}$$

The corresponding magnetic field can be found as

$$\mathbf{H} = \frac{\boldsymbol{\beta} \times \mathbf{E}}{\omega\mu} = \frac{\beta}{\omega\mu}(\hat{y}E_1 \mp j\hat{x}E_2)e^{-j\beta z} \tag{9.3.18}$$

Using the above, we find that the complex Poynting's vector as

$$\underset{\approx}{\mathbf{S}} = \mathbf{E} \times \mathbf{H}^* = \frac{\beta}{\omega\mu}\hat{z}(|E_1|^2 + |E_2|^2) \tag{9.3.19}$$

Then the time-average power density is

$$\langle \mathbf{S} \rangle = \frac{1}{2}\Re e\underset{\approx}{\mathbf{S}} = \frac{1}{2\eta}\hat{z}(|E_1|^2 + |E_2|^2) \tag{9.3.20}$$

When $E_1 = E_2 = E_0$, the above becomes

$$\langle \mathbf{S} \rangle = \frac{1}{2}\Re e\underset{\approx}{\mathbf{S}} = \frac{1}{\eta}\hat{z}|E_0|^2 \tag{9.3.21}$$

which is the same as in (9.3.14).

When $E_2 = 0$ for a linearly polarized wave, and $E_1 = E_0$, we have

$$\langle \mathbf{S} \rangle = \frac{1}{2}\Re e\underset{\approx}{\mathbf{S}} = \frac{1}{2\eta}\hat{z}|E_0|^2 \tag{9.3.22}$$

This is the same as what we have found before in (9.3.3). Notice that the Poynting's vector is a constant independent of z. This is because there is no reactive power in a plane wave of any polarization: the stored energy in the plane wave cannot be returned to the source!